Noncommutative Algebraic Equations and Noncommutative Eigenvalue Problem.

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February 1, 2008

Abstract

We analyze the perturbation series for noncommutative eigenvalue problem $AX = X\lambda$ where λ is an element of a noncommutative ring, A is a matrix and X is a column vector with entries from this ring. As a corollary we obtain a theorem about the structure of perturbation series for Tr x^r where x is a solution of noncommutative algebraic equation (for $r = 1$ this theorem was proved by Aschieri, Brace, Morariu, and Zumino, hep-th/0003228, and used to study Born-Infeld lagrangian for the gauge group $U(1)^k$).

We use the term "noncommutative algebraic equation" for the equation of the form

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n \qquad (1)$$

where the coefficients $a_1, ..., a_n$ and the unknown x belong to an associative (but not necessarily commutative) ring $\mathcal{A}$. It was shown in [1] that one can prove a generalization of Vieta theorem for the roots of (1). For example, if $x_1, ..., x_n$ are roots of (1), that are independent in some sense, we have $\mathrm{Tr} x_1 + ... + \mathrm{Tr} x_n = \mathrm{Tr} a_1$. Here trace is defined as an arbitrary linear functional on $\mathcal{A}$ obeying $\mathrm{Tr} ab = \mathrm{Tr} ba$ for all $a, b \in \mathcal{A}$.

Another proof of generalized Vieta theorem was given in [2]. This proof is based on the remark that the equation (1) is related to

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"noncommutative eigenvalue problem":

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= x_1\lambda \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= x_n\lambda
\end{aligned}
\tag{2}
$$

where the coefficients a_{ij}, unknowns x_i and "noncommutative eigen-value" λ are elements of associative ring $\mathcal{A}$. (It is assumed that the "noncommutative eigenvector" with entries x_i does not vanish.) It was shown in [2] that the problem (2) arises very naturally in the analysis of linear systems of first order differential equations in the ring $\mathcal{A}$ (and that (1) is related to higher order linear differential equations in $\mathcal{A}$).

Noncommutative Vieta theorem can be considered as a part of general theory of noncommutative functions. Noncommutative functions were studied in important series of papers [8]-[14]. In particular, these papers contain new proofs of noncommutative Vieta theorem ([12],[14],[8]). More precisely, the Gelfand- Retakh form of Vieta theorem is somewhat stronger than the statement of [1].

Recently the equation (1) was studied in the framework of perturbation theory in [3]. The authors of [3] prove some unexpected properties of perturbation series, that were conjectured for the case $n = 2$ in [4], [5]. (The equation (1) for $n = 2$ appears in the study of so called Born-Infeld lagrangian for the gauge group $U(1)^k$.)

The main goal of present letter is to state some general results about problems (1) and (2). We use these results to give a simple proof of the theorem of [3] and to generalize this theorem.

Let us present the system (2) in the form

$$
AX = X\lambda \tag{3}
$$

where $A \in \mathrm{Mat}_{n\times n}(\mathcal{A})$ and $X \in \mathrm{Mat}_{n\times 1}(\mathcal{A})$ (i. e. A is an $n \times n$ matrix, X is a column vector, both A and X have entries from $\mathcal{A}$). We will assume that $\mathcal{A}$ is a unital algebra over complex numbers and that

$$
A = \epsilon B + P \cdot 1
$$

where $\epsilon \in \mathbf{C}$, $B \in \mathrm{Mat}_{n\times n}(\mathcal{A})$ and $P \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ is an $n \times n$ matrix with entries from $\mathbf{C}$. We will consider the case, when P has n distinct eigenvalues $\kappa_1, ..., \kappa_n \in \mathbf{C}$. In this case we can find n solutions to the system (2) in the framework of perturbation theory

with respect to ϵ. More precisely, these solutions are formal power series

$$\lambda = \lambda^{(0)} + \epsilon\lambda^{(1)} + ... + \epsilon^n\lambda^{(n)} + ... \tag{4}$$

$$X = X^{(0)} + \epsilon X^{(1)} + ... + \epsilon^n X^{(n)} + ... \tag{5}$$

obeying (3). Using standard arguments one can show that there exist n such series for λ

$$^{(i)}\lambda = \kappa_i + \text{higher order terms.}$$

We denote corresponding X by $^{(i)}X$ (there is some freedom in the choice of the "eigenvector" $^{(i)}X$; we fix the choice in some way.) If $\mathcal{A}$ is a Banach algebra one can check that these series converge for sufficiently small ϵ.

Let us consider a matrix Ξ having "eigenvectors" $^{(1)}X, ...,^{(n)}X$ as its columns. It is easy to check that

$$A\Xi = \Xi\Lambda \tag{6}$$

where Λ is a diagonal matrix with entries $^{(1)}\lambda, ...,^{(n)}\lambda$. The equation (6) was used in [2] to obtain the information about noncommutative eigenvalues under the assumption that the matrix Ξ is invertible. We are working in the framework of perturbation theory, therefore Ξ is always invertible. (Ξ^{-1} exists as a series with respect to ϵ, because the series for Ξ starts with an invertible matrix.) We can say that

$$\Xi^{-1}A\Xi = \Lambda \tag{7}$$

It follows from (7) that

$$\Xi^{-1}(\oint_\Gamma A(A-\zeta)^{-1}d\zeta)\Xi = \oint_\Gamma \Lambda(\Lambda-\zeta)^{-1}d\zeta \tag{8}$$

Here Γ is an arbitrary curve on $\mathbf{C}$ that does not contain $\kappa_1, ..., \kappa_n$. The condition $\kappa_i \notin \Gamma$ permits us to say that $\Lambda - \zeta$ and therefore $A - \zeta$ are invertible in the framework of perturbation theory.

Let us assume now that Γ is a closed curve and the domain D, bounded by Γ, contains only one of the points $\kappa_1, ..., \kappa_n$. Then it follows from (8) that

$$\text{Tr}^{(i)}\lambda = (2\pi i)^{-1}\oint_\Gamma \text{Tr}A(A-\zeta)^{-1}d\zeta = (2\pi i)^{-1}\oint_\Gamma \text{Tr}\zeta(A-\zeta)^{-1}d\zeta \tag{9}$$

(Here we assumed that $\kappa_i \in D$). As earlier Tr stands for an arbitrary trace on $\mathcal{A}$; we used the relation $\mathrm{Tr}\Xi^{-1}K\Xi = \mathrm{Tr}K$.) Using the formula

$$(A - \zeta)^{-1} = (\epsilon B + P - \zeta)^{-1} = (1 + (P - \zeta)^{-1}\epsilon B)^{-1}(P - \zeta)^{-1}$$

$$= (P-\zeta)^{-1}-(P-\zeta)^{-1}\epsilon B(P-\zeta)^{-1}+(P-\zeta)^{-1}\epsilon B(P-\zeta)^{-1}\epsilon B(P-\zeta)^{-1}+.....$$

we can easily obtain the perturbation series for $\mathrm{Tr}^{(i)}\lambda$. However, as in standard perturbation theory (see [6]) it is more convenient to rearrange this series using the relation

$$\mathrm{Tr}\frac{d}{d\zeta}(BR(\zeta))^p = p\mathrm{Tr}R(\zeta)(BR(\zeta))^p. \qquad (10)$$

We introduced here the notation

$$R(\zeta) = (P - \zeta)^{-1},$$

the relation (10) follows from $dR/d\zeta = R(\zeta)^2$.

We get the following perturbation series for the trace of noncommutative eigenvalue $^{(i)}\lambda$:

$$\mathrm{Tr}^{(i)}\lambda = \kappa_i + \frac{1}{2\pi i}\sum_{p=1}\epsilon^p\frac{(-1)^p}{p}\oint_\Gamma \mathrm{Tr}(BR(\zeta)...BR(\zeta))d\zeta \qquad (11)$$

where Γ is a closed curve that encircles κ_i. The above consideration can be generalized to obtain an expression for $\mathrm{Tr}(^{(i)}\lambda)^r$. Namely, modifying slightly the derivation of (9) and (11) we obtain

$$\mathrm{Tr}(^{(i)}\lambda)^r = (2\pi i)^{-1}\oint_\Gamma \mathrm{Tr}A^r(A-\zeta)^{-1}d\zeta = (2\pi i)^{-1}\oint_\Gamma \zeta^r\mathrm{Tr}(A-\zeta)^{-1}d\zeta$$

$$= \kappa_i^r + \frac{1}{2\pi i}\sum_{p=1}\frac{\epsilon^p(-1)^p r}{p}\oint_\Gamma \zeta^{r-1}\mathrm{Tr}BR(\zeta)...BR(\zeta)d\zeta. \qquad (12)$$

The formula we obtained is a generalization of well known formula (see for example [6], p.79). Recall that the trace in (12) is an arbitrary linear functional on $\mathcal{A}$ that vanishes on all commutators, i.e. a linear functional on $\bar{\mathcal{A}} = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$. It is possible (and sometimes more convenient) to consider Tr in (12) as a natural map $\mathcal{A} \to \bar{\mathcal{A}}$.

We can consider entries of the matrix B as generators of free associative algebra. An element of free unital associative algebra $\mathcal{F}$ with generators $e_1, ..., e_n$ can be regarded as linear combination of expressions of the form $e_{\alpha_1}, ..., e_{\alpha_p}$, $p \geq 0$ (a linear combination of words with letters e_i). An element of $\mathcal{F}/[\mathcal{F}, \mathcal{F}]$ can be considered as linear combination of cyclic words. (The group $\mathbf{Z}_p$ acts on the set of words of length p by means of cyclic permutations. Two words belonging to the same orbit are equal mod $[\mathcal{F}, \mathcal{F}]$. This means that every word ω is equal mod $[\mathcal{F}, \mathcal{F}]$ to a cyclic word $\hat{\omega}$, i.e. to an average of all words in its $\mathbf{Z}_p$-orbit. We identify $\mathcal{F}/[\mathcal{F}, \mathcal{F}]$ with the subspace of $\mathcal{F}$ spanned by all cyclic words.) Using (12) we can express Tr $^{(i)}\lambda$ as a linear combination of cyclic words. A cyclic word $\hat{\omega}$ where $\omega = b_{\alpha_1, \beta_1} b_{\alpha_2, \beta_2}...b_{\alpha_p, \beta_p}$ enters this combination with coefficient

$$c(\omega) = c_{\alpha_1, \beta_1, ..., \alpha_p, \beta_p} = \frac{(-1)^p \epsilon^p}{2\pi i} \oint_\Gamma \mathrm{Tr}(R_{\beta_1, \alpha_2}(\zeta)...R_{\beta_{p-1}, \alpha_p}(\zeta) R_{\beta_p, \alpha_1}(\zeta)) d\zeta.$$
(13)

The analogous coefficient in the expression for Tr $(^{(i)}\lambda)^r$ looks as follows:

$$c^{(r)}(\omega) = (c)^r_{\alpha_1, \beta_1, ..., \alpha_p, \beta_p} = \frac{(-1)^p \epsilon^p r}{2\pi i} \oint_\Gamma \zeta^{r-1} \mathrm{Tr}(R_{\beta_1, \alpha_2}(\zeta)...R_{\beta_p, \alpha_1}(\zeta)) d\zeta.$$
(14)

Let us come back to the equation (1). For every solution x of this equation we can construct a solution of eigenvalue problem (2) with

$$A = \begin{pmatrix} a_1 & . & . & . & a_n \\ 1 & & & & \\ & . & & & \\ & & . & & \\ & & & 1 & \end{pmatrix}$$
(15)

taking $\lambda = x$, $x_k = x^{n-k}$. This remark, that was used in [2], allows us to obtain information about solution of (1) from the information about eigenvalue problem. We want to study (1) in the framework of perturbation theory; therefore we will write (1) in the form

$$x^n = \epsilon a_1 x^{n-1} + ... + \epsilon a_n + 1$$
(16)

Then we have n perturbative solutions that correspond to n solutions of "unperturbed" equation. We can replace equation (16) with

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the eigenvalue problem (3) with the matrix

$$A = \epsilon B + P \cdot 1 = \epsilon \begin{pmatrix} a_1 & \cdots & a_n \\ 0 & & 0 \\ & & \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ 1 & & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & 1 & 0 \end{pmatrix} \qquad (17)$$

(Every perturbative solution $x(\epsilon)$ of (16) gives a perturbative solution of eigenvalue problem. We obtain one-to-one correspondence because both problems have precisely n perturbative solutions.)Now using (11) we get an explicit expression for the traces of roots of (16). To apply (11) we should calculate $R(\zeta) = (P-\zeta)^{-1}$; it is convenient to express it in the form

$$R(\zeta) = (P - \zeta)^{-1} = -\zeta^{-1}(1 - \zeta^{-1}P)^{-1} = \sum_{\alpha=0}^{n-1} \frac{\zeta^{n-\alpha-1}}{1-\zeta^n} P^\alpha.$$

(We used that $P^n = 1$). We see that

$$\begin{aligned} R_{\alpha\beta}(\zeta) &= \frac{\zeta^{n-1-(\alpha-\beta)}}{1-\zeta^n} \quad \text{if} \quad \alpha \geq \beta \\ R_{\alpha\beta}(\zeta) &= \frac{\zeta^{-1-(\alpha-\beta)}}{1-\zeta^n} \quad \text{if} \quad \alpha < \beta \end{aligned} \qquad (18)$$

It is easy to verify using (13) that the cyclic word $\hat{\omega}$ where $\omega = a_{\alpha_1}...a_{\alpha_p}$ enters the perturbative expression for the trace of a root $x(\epsilon)$ of (16) with the coefficient

$$c(\omega) = c_{\alpha_1,...,\alpha_p} = \frac{(-1)^p \epsilon^p}{2\pi i} \oint_\Gamma \frac{\zeta^{p(n-1)-\sum \alpha_i}}{(1-\zeta^n)^p} d\zeta. \qquad (19)$$

Corresponding coefficients for the trace of $x(\epsilon)^r$ can be obtained from (14):

$$c^{(r)}(\omega) = c^{(r)}_{\alpha_1,...,\alpha_p} = \frac{(-1)^p \epsilon^p r}{2\pi i} \oint_\Gamma \frac{\zeta^{r+p(n-1)-\sum \alpha_i}}{(1-\zeta^n)^p} d\zeta. \qquad (20)$$

We obtain that $c^{(r)}_{\alpha_1,...,\alpha_p}$ is symmetric with respect to $\alpha_1, ..., \alpha_p$. This statement was proved in [3] for $r = 1$. It permits us to obtain a perturbative expression for $\operatorname{Tr} x(\epsilon)^r$ from the solution of corresponding commutative problem.

Namely, we can construct a linear map $\mathcal{S}$ of commutative algebra of power series with respect to $a_1, ..., a_n$ into corresponding noncommutative object assigning to every monomial a sum of all words corresponding to this monomial multiplied by a normalization factor. (There exists a natural homomorphism π of noncommutative free algebra into commutative polynomial algebra. The word ω corresponds to a monomial ρ if $\pi(\omega) = \rho$. The normalization factor is determined by the condition $\pi(\mathcal{S}(\rho)) = \rho$.)

The following statement (proved in [3] for $r = 1$) is an immediate consequence of the symmetry of coefficients $c^{(r)}_{\alpha_1,...,\alpha_p}$.

Let $x_{comm}(\epsilon)$ be a perturbative solution of ordinary algebraic equation $x^n = \epsilon(a_1 x^{n-1} + ... + a_n) + 1$. Then $\mathrm{Tr}\,(x(\epsilon)^r) = \mathcal{S}(x_{comm}(\epsilon)^r)$.

The integral

$$\gamma^\rho_{n,p} = \oint_\Gamma \frac{\sigma^\rho}{(1 - \sigma^n)^p} d\sigma \tag{21}$$

can be easily calculated [7]. One can use for example recursion formulas

$$\gamma^\rho_{n,p} = \frac{n(1 - p) + \rho + 1}{n(1 - p)} \gamma^\rho_{n,p-1} \tag{22}$$

and

$$\gamma^\rho_{n,p} = \frac{\rho + 1 - n}{\rho + 1 - np} \gamma^{\rho-n}_{n,p}. \tag{23}$$

Using the notation (21) we can represent the coefficients $c^{(r)}_{\alpha_1,...,\alpha_p}$ in the form

$$c^{(r)}_{\alpha_1,...,\alpha_p} = \frac{(-1)^p \epsilon^p r}{2\pi i} \gamma^{r+p(n-1)-\sum \alpha_i}_{n,p}. \tag{24}$$

We will use the formula (24) to derive some statements about eigenvalues of matrix $\epsilon B + P \cdot 1$ where

$$P = \begin{pmatrix} 0 & . & . & . & 1 \\ 1 & & & & \\ & . & & & \\ & & . & & \\ & & & 1 & 0 \end{pmatrix} \tag{25}$$

but B is an arbitrary matrix. In this case using (20) and (21) we obtain the following expression for the coefficient $c^{(r)}(\omega) = c^{(r)}_{\alpha_1,\beta_1,...,\alpha_p,\beta_p}$:

$$c^{(r)}_{\alpha_1,\beta_1,...,\alpha_p,\beta_p} = \frac{(-1)^p \epsilon^p r}{2\pi i} \gamma^{r+\sum \alpha_i - \sum \beta_i - p + Kn}_{n,p} \tag{26}$$

where K is the number of indices i obeying $\beta_i \geq \alpha_{i+1}$ (we identify α_{p+1} with α_1). Let ω' be a word corresponding to the same monomial as $\omega = b_{\alpha_1\beta_1}...b_{\alpha_p\beta_p}$ (i.e. ω' is obtained from ω by means of permutation of factors $b_{\alpha_i\beta_i}$). It is clear that the expression for $c(\omega')$ is almost identical to the expression for $c(\omega)$; only K changes. We can use (23) to find $c(\omega')/c(\omega)$; if $K(\omega') - K(\omega) = s \geq 0$ we obtain

$$\frac{c^{(r)}(\omega')}{c^{(r)}(\omega)} = \frac{\rho + 1 - n}{\rho + 1 - np} \cdot \frac{\rho + 1 - 2n}{\rho + 1 - n(p+1)} \cdot \dots \cdot \frac{\rho + 1 - ns}{\rho + 1 - n(p+s)} \quad (27)$$

where $\rho = \sum \alpha_i - \sum \beta_i - p + r$. It is easy to check that using (27) we can find the perturbative expression for the trace of noncommutative eigenvalue if we know the solution of corresponding commutative problem. However, it seems that the explicit expression (26) is more convenient.

Acknowledgments

I am indebted to D.Brace, A.Chorin,I.Gelfand, M. Kapranov, B. Morariu,V. Retakh and B.Zumino for useful discussions.

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